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LETTER TO THE EDITOR

Small g and large λ solution of the Schrödinger equation for the interaction $\lambda x^2/(1+gx^2)$

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Abstract. Using perturbation theory, asymptotic expansions are derived for the eigenenergies and eigenfunctions of the wave equation for the interaction $\lambda x^2/(1+gx^2)$ in the range of small values of g and large values of λ. The first few energy eigenvalues are calculated and found to be comparable with the non-perturbative results obtained recently by A K Mitra.

The solution of the one-dimensional Schrödinger equation with a potential of the type

$$V(x)=x^2+\lambda x^2/(1+gx^2) \tag{1}$$

is required in different contexts. As pointed out by Biswas *et al* (1973), the Schrödinger equation with an interaction Lagrangian of the type $\lambda x^2/(1+gx^2)$ becomes analogous to a zero-dimensional field theory with a non-linear Lagrangian. Another area where the solution of such a problem is required, as emphasised by Mitra (1978), is in laser theory, particularly when one deals with specific models. Furthermore, the reduction of the Fokker–Planck equation of a single-mode laser may also lead (see for example Haken 1970, Risken and Vollmer 1967) to such a Schrödinger equation under certain conditions.

Recently Mitra (1978) has calculated the ground state and the first two excited state energy eigenvalues for this interaction by using the Ritz variational method in combination with the Givens–Householder algorithm. As there is a use of various quadrature formulae in his method, it involves a considerable amount of computer time, which becomes even longer if one also computes the eigenfunctions. In the present note, we solve this problem perturbatively and find that over a large range of h $(=g/2(1+\lambda)^{1/2})$, large λ asymptotic expansions also offer a good degree of accuracy.

Here we use the perturbation procedure employed by Müller (1970) (and more recently by Aly *et al* 1975) in a study of the Gauss potential. For the sake of completeness only the essential steps of the method are outlined. We start with the one-dimensional Schrödinger equation

$$[-\mathrm{d}^2/\mathrm{d}x^2+V(x)]\psi=E\psi$$

with the potential (1), which, for $gx^2<1$ $(g>0)$, expressed as

$$V(x)=(1+\lambda x^2)+\sum_{i=1}^{\infty}(-g)^i\lambda x^{2(i+1)} \tag{1$'$}$$

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leads to the form

$$\left[\frac{d^2}{dZ^2}+\frac{k^2}{2(1+\lambda)^{1/2}}-\frac{1}{4}Z^2\right]\psi=-\frac{\lambda}{4(1+\lambda)}\sum_{i=2}^{\infty}h^{i-1}(-Z^2)^i\psi \tag{2}$$

where we have set

$$Z^2=2(1+\lambda)^{1/2}x^2;\qquad h=g/2(1+\lambda)^{1/2};\qquad k^2=E;\qquad 2m=\hbar=1. \tag{3}$$

The expansion (1′) implies that we restrict ourselves to the domain $gx^2<1$. Our solution will therefore be that branch of the solution which is valid in this region. In other regions we have (Müller-Kirsten and Bose 1978, Müller-Kirsten *et al* 1978, Kaushal and Müller-Kirsten 1979) different branches (such as the WKB solutions) which can be shown to be analytically continuable to our branch (in analogy to the considerations in these works). These various pieces, however, are all associated with one and the same eigenvalue. Since we are here interested primarily in the eigenvalues, we do not discuss the other type of solutions and their matching.

To the lowest order in h (when $g\to 0$) we have

$$\left(\frac{d^2}{dZ^2}+\frac{1}{2}q-\frac{1}{4}Z^2\right)\psi_q^{(0)}=0,\qquad q=k^2/(1+\lambda)^{1/2}$$

where $\psi_q^{(0)}(Z)=D_{1/2(q-1)}(Z)$ is a parabolic cylindrical function. The square integrability of $\psi_q^{(0)}$ demands that q be an odd integer, i.e. $q=2n+1$, with $n=0,1,2\ldots.$ In equation (2) we now set

$$k^2/(1+\lambda)^{1/2}=q+h\Delta \tag{4}$$

where $h\Delta$ is of $O(h)$.

Substituting equation (4) into equation (2) we obtain

$$\mathscr{D}_q\psi=h\{\Delta+[\lambda/2(1+\lambda)]\sum_{i=2}^{\infty}h^{i-2}(-Z^2)^i\}\psi \tag{5}$$

with $\mathscr{D}_q=-[2\,d^2/dZ^2+q-\frac{1}{2}Z^2]$, which in the lowest order satisfies $\mathscr{D}_q\psi_q^{(0)}=0$. Now $\psi_q^{(0)}$ (or $D_{\frac{1}{2}(q-1)}$) satisfies the recurrence relation

$$Z\psi_q=\psi_{q+2}+\tfrac{1}{2}(q-1)\psi_{q-2} \tag{6a}$$

or

$$Z^2\psi_q=(q,q+4)\psi_{q+4}+(q,q)\psi_q+(q,q-4)\psi_{q-4} \tag{6b}$$

with

$$(q,q+4)=1,\qquad (q,q)=q,\qquad (q,q-4)=\tfrac{1}{4}(q-1)(q-3).$$

In general we write

$$Z^{2m}\psi_q=\sum_{j=-m}^{+m}S_m(q,q+4j)\psi_{q+4j} \tag{7}$$

where the coefficients $S_m(q,q+4j)$ can be calculated from the repeated use of equation

(6*b*). The lowest-order approximation leaves uncompensated the contribution

$$R_q^{(0)}=h\left[\Delta+\frac{\lambda}{2(1+\lambda)}\sum_{i=0}^{\infty}(-h)^i Z^{2(i+2)}\right]\psi_q(Z)$$
$$=\sum_{i=0}^{\infty}h^{i+1}\sum_{j=-(i+2)}^{i+2}[q,q+4j]_{i+1}\psi_{q+4j} \tag{8}$$

where

$$\left.\begin{array}{ll}[q,q]_1=\Delta+[\lambda/2(1+\lambda)]S_2(q,q) & \text{for } i=0, j=0\\ [q,q+4j]_{i+1}=[(-1)^i\lambda/2(1+\lambda)]S_{i+2}(q,q+4j) & \text{otherwise}\end{array}\right\}. \tag{9}$$

Following the procedure discussed by Müller (1970), the first-order correction to the wavefunction is given by

$$\psi_q^{(1)}=\sum_{i=0}^{\infty}h^{i+1}\sum_{\substack{j=-(i+2)\\ j\neq 0}}^{i+2}\frac{[q,q+4j]_{i+1}}{4j}\psi_{q+4j} \tag{10}$$

provided $[q,q]_1=0$. Again, in the first order the uncompensated terms left are given by

$$R_q^{(1)}=\sum_{i=0}^{\infty}h^{i+1}\sum_{\substack{j=-(i+2)\\ j\neq 0}}^{i+2}\frac{[q,q+4j]_{i+1}}{4j}R_{q+4j}^{(0)}. \tag{11}$$

The higher-order corrections to the eigenfunctions, i.e. $\psi_q^{(2)}, \psi_q^{(3)}\ldots$ are now obtained in a manner analogous to the derivation of $\psi_q^{(1)}$. Then adding successive contributions we obtain

$$\psi_q=\psi_q^{(0)}+\psi_q^{(1)}+\psi_q^{(2)}+\ldots$$

which apart from a normalisation constant can be written as

$$\psi_q=\sum_{i=0}^{\infty}h^{i+1}\sum_{j=-(i+2)}^{(i+2)}C_i(q,j)\psi_{q+4j}(Z) \tag{12}$$

where the coefficients $C_i(q,j)$ follow by comparison. For equation (12) to be a solution of equation (5), the sum of the coefficients of ψ_q in $R_q^{(0)}, R_q^{(1)}\ldots$—left uncompensated so far—must be set equal to zero, i.e.

$$0=h[q,q]_1+h^2\left\{[q,q]_2+\sum_{\substack{j=-2\\ j\neq 0}}^{+2}\frac{[q,q+4j]_1}{4j}[q+4j,q]_1\right\}$$
$$+h^3\left\{[q,q]_3+\sum_{\substack{j=-2\\ j\neq 0}}^{+2}\frac{[q,q+4j]_1}{4j}[q+4j,q]_2+\sum_{\substack{j=-3\\ j\neq 0}}^{+3}\frac{[q,q+4j]_2}{4j}[q+4j,q]_1\right.$$
$$\left.+\sum_{\substack{j=-2\\ j\neq 0}}^{+2}\frac{[q,q+4j]_1}{4j}\sum_{\substack{j'=-2\\ j+j'\neq 0}}^{+2}\frac{[q+4j,q+4j+4j']_1}{4j+4j'}[q+4j+4j',q]_1\right\}+\ldots \tag{13}$$

where the terms are arranged in the powers of h. This is the equation which determines Δ and hence the eigenvalues E_q. The coefficients $[q,q+4j]_{i+1}$ can be evaluated with the help of equations (6*b*), (7) and (9). Then

$$\begin{aligned}E \equiv E_q = {} & q(1+\lambda)^{1/2} - \frac{3\lambda(q^2+1)}{4(1+\lambda)^{1/2}}h + \frac{\lambda q}{16(1+\lambda)^{3/2}}[(q^2+5)(20+3\lambda)+18\lambda]h^2 \\ & -([35\lambda/16(1+\lambda)^{1/2}](q^4+14q^2+g) \\ & +[\lambda^2/16384(1+\lambda)^{5/2}][(q-1)(q-3) \\ & \times\{192(1+\lambda)(21q^3-136q^2+383q-420)+\lambda[(q-5)(q-7) \\ & \times(35q^2-304q+579)+128(q-2)(q^3-18q^2+44q-54)]\} \\ & -(q+1)(q+3)\{192(1+\lambda)(21q^3+136q^2+383q+420) \\ & -\lambda[(q+5)(q+7)(35q^2+304q+579)+192(q+2)^2(q^2+8q+17)]\}]) \\ & \times h^3 + \mathrm{O}(h^4). \end{aligned} \tag{14}$$

The expressions (12) and (14), representing the asymptotic expansions for the large values of λ, are our main results. The solution (12) is valid for $h<1$ and $|Z| \leqslant \mathrm{O}(h^{-1/2})$. While the first condition is obvious, the latter is obtained (Kaushal amd Müller-Kirsten 1979) by demanding that the ratio of the $(i+1)$th and ith terms in equation (12) decrease at large Z and for $h \to 0$. In fact, the second condition, expressing the domain of validity in terms of the size of h, turns out to be consistent with $gx^2 < 1$. Although it is not difficult to compute the eigenfunctions from equation (12), we evaluate here the terms in equation (14) only up to $\mathrm{O}(h^3)$. To check the quality of the approximation involved in these terms, we have calculated the first four energy eigenvalues for different sets of λ and g. The results are shown in table 1, for $q = 1, 3, 5$ and 7. For small g ($g \leqslant 0{\cdot}2$), as can be seen from this table, the results are comparable (up to the third decimal place) to those of Mitra (1978). However, for large g and small λ, it is found that equation (14) underestimates the eigenenergies.

On the other hand, it may be noted that equation (14) is an asymptotic expansion for large λ and alternates in sign. For such cases there exists (Dingle 1973) a criterion to obtain an asymptotically correct value of the sum by terminating the power series. For this purpose the expansion is terminated at the least term (least in magnitude) and the last term is multiplied by $\frac{1}{2}$. Using this criterion we find a considerable improvement over the results of equation (14) for large g. The results corresponding to $g = 0{\cdot}5$, $0{\cdot}8$ and $1{\cdot}0$ in table 1 are given after accounting for such a correction. Further improvement in these results is possible by using Dingle's convergence factors (Dingle 1973). However, such corrections are rather involved and may be estimated when one applies the potential (1) to a specific problem.

Finally, it may be mentioned that with a little modification the results of this paper can be applied to anharmonic perturbations of the type

$$\sum_{i=2}^{m} a^i x^{2i}, \qquad \sum_{i=2}^{m} (-a)^i x^{2i}; \qquad ax^{2n}; \qquad ax^{2n+1}$$

where a is taken to be small (<1) and $n = 2, 3, 4 \ldots$. In the last two cases one has to proceed with a single term on the right-hand side of equation (2).

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Table 1. The first four energy eigenvalues obtained from the asymptotic expansion (14) for different values of λ and g. The numbers in the bracket correspond to the non-perturbative results of Mitra (1978).

g	q \ λ	0·1	0·2	0·5	1·0	2·0	5·0	10·0	20·0	50·0	100
0·1	1	1·04305	1·08479	1·20290	1·38045	1·68557	2·38951	3·250244	4·512411	7·068692	9·9761778
		(1·04317)	(1·08495)	(1·20303)	(1·38053)	(1·68561)	(2·38954)	(3·25026)	(4·51242)	(7·06869)	(9·97618)
	3	3·1189	3·2354	3·5695	4·0789	4·9676	7·0501	9·61843	13·39698	21·06057	29·78110
		(3·12008)	(3·23700)	(3·57080)	(4·0798)	(4·96859)	(7·05096)	(9·61906)	(13·39736)	(21·06073)	(29·78119)
	5	5·175	5·351	5·864	6·661	8·078	11·4576	15·7228	22·0016	34·7621	49·29177
		(5·18109)	(5·35866)	(5·87158)	(6·667)	(8·08680)	(11·48480)	(15·72933)	(22·00557)	(34·76383)	(49·29269)
	7	7·208	7·430	8·093	9·132	11·012	15·656	21·554	30·321	48·1716	68·5079
0·2	1	1·0376	1·0751	1·1843	1·3519	1·6448	2·3341	3·18719	4·44463	6·99748	9·903570
	3	3·088	3·185	3·481	3·945	4·772	6·775	9·3027	13·0571	20·7039	29·41767
	5	5·072	5·194	5·628	6·324	7·572	10·746	14·888	21·109	33·831	48·3458
	7	7·171	7·352	7·555	8·407	9·921	14·104	19·835	28·536	46·354	66·677
0·5	1	1·032	1·064	1·103	1·263	1·537	2·187	3·0139	4·2535	6·7922	9·69185
		(1·03121)	(1·06196)	(1·15156)	(1·29295)	(1·55104)	(2·19211)	(3·01685)	(4·25506)	(6·79278)	(9·69215)
	3	3·061	3·130	3·362	3·459	4·225	6·006	8·400	12·075	19·663	28·3514
		(3·07389)	(3·14722)	(3·36380)	(3·71390)	(4·37658)	(6·12105)	(8·482)	(12·12361)	(19·68503)	(28·8362)
	5	5·022	5·071	5·311	5·852	5·655	8·222	12·560	18·301	31·016	45·521
		(5·09305)	(5·18591)	(5·46320)	(5·92063)	(6·81529)	(9·32076)	(12·948)	(18·79614)	(31·23804)	(45·636)
	7	6·916	6·887	7·011	7·556	8·400	10·869	15·675	23·783	41·170	61·321
0·8	1	1·022	1·045	1·125	1·252	1·410	2·051	2·856	4·076	6·5974	9·4883
	3	3·010	3·036	3·147	3·493	3·416	5·159	7·463	11·088	18·641	27·307
	5	4·890	4·827	4·824	5·121	6·061	7·363	10·539	16·167	28·464	42·597
	7	6·660	6·418	6·073	6·150	7·125	8·576	12·092	19·413	36·234	56·149
1·0	1	1·015	1·033	1·100	1·227	1·289	1·957	2·754	3·963	6·472	9·3567
		(1·02410)	(1·04801)	(1·11854)	(1·23235)	(1·44732)	(2·01300)	(2·78233)	(3·97769)	(6·47811)	(9·3594)
	3	2·976	2·974	3·049	3·305	3·899	4·444	6·754	10·390	17·952	26·615
		(3·05149)	(3·10281)	(3·25577)	(3·50738)	(3·99840)	(5·37944)	(7·41751)	(10·79063)	(18·12871)	(26·705)
	5	4·801	4·665	4·499	4·634	5·410	6·647	9·355	14·694	26·779	41·030
		(5·0344)	(5·11793)	(5·29488)	(5·58977)	(6·17848)	(7·92192)	(10·701)	(15·698)	(27·375)	(41·441)
	7	6·489	6·106	5·448	5·212	5·874	7·661	10·091	16·754	33·092	52·802

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